ORIGINAL PAPER

Non-Markovian dynamics and entanglement in quantum Brownian motion

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Abstract   Dynamical aspects of quantum Brownian motion in a low temperature environment are investigated. We give a systematic calculation of quantum entanglement among two Brownian oscillators without invoking Born–Markov approximation widely used for the study of open systems. Our approach is suitable to probe short time dynamics at cold temperatures where many experiments on quantum information processing are performed.

1 Introduction

Chemical systems such as liquid NMR have been actively used for quantum information processing experiments [1]. Hybrid devices based on molecular ensembles on solid state circuits provide many advantages as quantum processors such as long lasting memory, good controlability and scalability [2].

Implementing quantum information processing devices requires precise knowledge of quantum open systems. The detailed study of quantum open systems are often prevented by a limitation of analytical or numerical resources to probe systems with large degrees of freedom. The first principle calculation of open quantum systems based on projection operators [3,4] or influence functionals [5] has been developed and applied to numerous problems [6,7]. Reduced dynamics obtained by these methods follows non-Markovian evolution carrying the memory of environment. Solving the equations of motion obtained by these method directly is generally difficult to deal with as they are given by integro-differential equations.

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Various approximation schemes have been used to simplify the exact evolution to study open system evolution. Most commonly used approximation is Born approximation in which the system–bath interaction is treated perturbatively. However, naive Born approximation applied to quantum open systems does not guarantee the positivity of the density matrix and requires a caution particularly when evaluating quantities sensitive to this aspect such as quantum entanglement.

For a weak system–bath interaction, bath is disturbed by the interaction only in a small amount and tends to reset to the original state in a short time compared to the system time scale. The correlation among the bath variables may become negligible compared to the system time scale. In a Born–Markov approximation, the bath correlation is ignored. At sufficiently low temperature, however, the bath memory can persist for a long time and this approximation is expected to break down. Regardless of the temperature, the noise with long range correlation such as 1/f noise cannot be treated under this approximation. 1/f noise is ubiquitous causing a problem in implementing a quantum circuit using electronic instruments in solid state, ion trap, or hybrid molecular devices. Decoherence in a non-equilibrium bath [8] also shows a peculiar behavior which is significantly different from the one under Markov approximation.

Further approximation such as rotating-wave approximation is commonly used with Born–Markov approximation. There we assume weak system–bath interaction such that a coupling becomes only among near resonant modes. We further ignore counter rotating terms since they are expected to be averaged out during a sufficiently long observation time. The rotating-wave approximation is thus not suitable for describing short time dynamics in a strong coupling regime.

Quantum entanglement, manifestation of intrinsic non-locality in quantum mechanics [9], now became one of the most active research topics in quantum information science. It also has been a long historical issue as the discrepancy between quantum and classical mechanics is a serious obstacle to understand macroscopic classical and quantum mechanics from the view of microscopic quantum dynamics [10,11].

Conversely but for the same reason, to manipulate quantum systems in order to be useful in the macroscopic world is a formidable task. Recent progress in this direction is motivated by rapid progress in quantum technology, where there is a demand for a precise control of small quantum devices. Quantum entanglement is considered to be a valuable resource for practical applications such as quantum computation and communication [12].

Many works have been devoted to clarify the rigorous criteria for entanglement. For continuous variables, the necessary and sufficient criteria for entanglement can be given in terms of Peres-Horodecki criteria [13–15], negativity [16], and entanglement of formation [17]. Most of these works are devoted to study the static properties of general pure and mixed states. In light of information processing, it is also desirable to study the dynamical properties based on the realistic models in an open system setting.

Quantized harmonic oscillators have been playing an important role in a history of quantum mechanics. Realization of quantum protocols based on continuous variables such as quantum teleportation [18] and quantum key distribution [19] show that they also play the similar crucial role in quantum information science. Realization of the similar protocols based on solid state devices is a highly formidable task since the

disturbance from environment in solid state systems is much stronger than in optical systems.

So far the studies of continuous variables for open systems are based on the master equation under Born–Markov rotating wave approximation such as Lindblad equations [20] or the phenomenological model with high temperature Markovian bath [21]. Applications of these methods are limited to the high temperature regime or slow dynamics with Ohmic noise, while actual solid state implementation operates at low temperatures manipulated with fast pulses and often suffers from 1/f noise.

Most of the solid state implementation of quantum information devices operate in a ultra-cold temperature in order to maintain quantum coherence. For instance, typical superconducting qubits operate <100 mK [22]. At ultra-cold temperatures, the memory effect in the environment variables is no longer negligible. The system-environment interaction in solid state devices is typically much stronger than those qubits based on atom-optical devices. These conditions are opposite to those where the Born–Markov approximation is applicable. Thus we need to develop the method to probe the regime beyond the conventional Born–Markov approximation.

Born and rotating wave approximations are limited to weak coupling regime and do not guarantee the positivity of the density matrix evolution. Since separability criteria is based on the positivity of the density matrix, a caution is required to study entanglement under these approximations.

In the present paper, we study the precise mechanism of open system entanglement dynamics based on a Brownian oscillator model [23], where both system and environment consist of harmonic oscillators. Since our model is formally exactly solvable, it makes us possible to probe the precise dynamics of entanglement without conventional approximations. Previous works for the study of a two-level system revealed that non-Markovian dynamics that plays the major role at low temperatures is highly non-trivial [6,24].

In Sect. 2, we develop systematic tools for our analysis. The reduced density matrix in the Wigner representation is calculated for two quantum Brownian oscillator model. In Sect. 3, we first make sure that uncertainty relations are always satisfied in our model. They are time and temperature dependent as our system stays mostly far from equilibrium. Next we make partial transpose operation to our density matrix. In the Wigner representation, the partial transpose operation corresponds to the partial mirror reflection. We again study uncertainty relations after partial transpose. According to the Peres-Horodecki-Simon criteria, the violation of uncertainty relations after partial mirror reflection can be used as a signature of quantum entanglement. Entanglement measures, the negativity and the logarithmic negativity, are also calculated for comparison.

2 General formulation

We consider a system composed of two harmonic oscillators. Our Lagrangian is given by

$$L_S = \sum_{j=1}^{2} \frac{M_j \dot{R}_j^2}{2} - V(R_1, R_2) \,, \tag{1}$$

where the potential V is assumed to be harmonic:

$$V(R_1, R_2) = \sum_{j=1}^{2} \frac{M_j \Omega_j^2}{2} R_j^2. \tag{2}$$

Each oscillator variable R_j located at the origin $x = 0$ couples linearly with a scalar field ϕ via minimal coupling as

$$L_I = -\sum_{j=1}^{2} \lambda_j \dot{R}_j(t) \phi(0, t). \tag{3}$$

The field Lagrangian is

$$L_\phi = \frac{1}{2} \int dx \left[(\partial_x \phi)^2 - m^2 \phi^2 \right]. \tag{4}$$

The scalar field ϕ propagating in one-dimensional space allows a mode decomposition:

$$\phi(x, t) = \int \frac{dk}{(2\pi)^{1/2} \sqrt{2\omega_k}} \left\{ b_k e^{-i\omega_k t + ikx} + b_k^\dagger e^{i\omega_k t - ikx} \right\}. \tag{5}$$

In this paper, we study the massless field, then $\omega_k = |k|$. We consider the field ϕ as an environment and trace out to obtain dissipative dynamics for oscillator variables R_j.

The Heisenberg equations that R_j satisfy have the form of damped harmonic oscillators [7]:

$$M_j \frac{d^2 R_j(t)}{dt^2} + M_j \Omega_j^2 R_j(t) - 2 \sum_{l=1}^{2} \int_0^t ds \frac{d\alpha_{Ijl}(t, s)}{dt} \frac{dR_l(s)}{ds} = 0, \tag{6}$$

where

$$\alpha_{Ijl}(t, t') = -\lambda_j \lambda_l \sum_k \sin \left[\omega_k (t - t') \right] / \omega_k \tag{7}$$

is an imaginary part of the response function [5] defined as $\alpha_{jl}(t, t') \equiv 2\lambda_j \lambda_l \sum_k e^{-i\omega_k(t-t')}/\omega_k$ with $\sum_k \equiv \int dk/(2\pi)$. Note that $\alpha_{Ijl}(t, t')$ is antisymmetric in indices: $\alpha_{Ijl}(t, t') = -\alpha_{Ilj}(t', t)$. Without any approximations, Eq. 6 has a non-local form with kernels given by $\alpha_{Ijl}(t, t')$. Thus the value of R_j at each moment depends on their entire history of the past.

In one dimensional space when a ultraviolet cutoff of the field modes is brought to infinity, Eq. 6 will be reduced to a local form. We write $\gamma_1 \equiv \lambda_1^2/M_1$, $\gamma_2 \equiv \lambda_2^2/M_2$, $\gamma_{12} \equiv \lambda_1 \lambda_2/M_2$, $\gamma_{21} \equiv \lambda_1 \lambda_2/M_1$. In this case, Heisenberg equations of motion will be

$$M_1\ddot{R}_1(t) + M_1\Omega_1^2 R_1(t) + \gamma_1 M_1 \dot{R}_1(t) + \gamma_{12} M_2 \dot{R}_2(t) = 0$$
$$M_2\ddot{R}_2(t) + M_2\Omega_2^2 R_2(t) + \gamma_2 M_2 \dot{R}_2(t) + \gamma_{21} M_1 \dot{R}_1(t) = 0. \tag{8}$$

We write a pair of solutions of (8) with initial conditions $R_1(0) = R_2(0) = 0$ and $\dot{R}_1(0) = 1$, $\dot{R}_2(0) = 0$ as $h_1(t)$ and $h_3(t)$. For identical two oscillators ($M_1 = M_2 = 1$, $\Omega_1 = \Omega_2 \equiv \Omega$) coupled to ϕ with equal strength ($\lambda_1 = \lambda_2$) (hereafter we will drop indices from these parameters), the solutions are given by $h_1(t) \equiv (g_1(t) + g_0(t))$ and $h_3(t) \equiv (g_1(t) - g_0(t))/2$, where

$$g_1(t) = \frac{\sin(\Omega_r t)}{\Omega_r} e^{-\gamma t} \quad \text{and} \quad g_0(t) = \frac{\sin(\Omega t)}{\Omega} \tag{9}$$

are solutions corresponding to two normal modes of a coupled oscillator and $\Omega_r^2 \equiv \Omega^2 - \gamma^2$.

General solutions with arbitrary initial conditions R_{j0} and P_{j0} of coupled Heisenberg Eq. 8 for $j = 1, 2$ are

$$R_j(t) = C_{R_j R_1} R_{10} + C_{R_j P_1} P_{10} + C_{R_j R_2} R_{20} + C_{R_j P_2} P_{20}$$
$$+ \lambda \int_0^t ds\, g_1(t - s)\dot{\phi}(s),$$
$$P_j(t) = C_{P_j R_1} R_{10} + C_{P_j P_1} P_{10} + C_{P_j R_2} R_{20} + C_{P_j P_2} P_{20}$$
$$+ \lambda \int_0^t ds\, g_2(t - s)\dot{\phi}(s), \tag{10}$$

where $g_2 \equiv \dot{g}_1$. The expectation value of phase space variables can be expressed in a matrix form:

$$\begin{pmatrix} \langle R_1 \rangle \\ \langle P_1 \rangle \\ \langle R_2 \rangle \\ \langle P_2 \rangle \end{pmatrix} = \mathcal{C} \begin{pmatrix} R_{10} \\ P_{10} \\ R_{20} \\ P_{20} \end{pmatrix} = \begin{pmatrix} C_{R_1 R_1} & C_{R_1 P_1} & C_{R_1 R_2} & C_{R_1 P_2} \\ C_{P_1 R_1} & C_{P_1 P_1} & C_{P_1 R_2} & C_{P_1 P_2} \\ C_{R_2 R_1} & C_{R_2 P_1} & C_{R_2 R_2} & C_{R_2 P_2} \\ C_{P_2 R_1} & C_{P_2 P_1} & C_{P_2 R_2} & C_{P_2 P_2} \end{pmatrix} \begin{pmatrix} R_{10} \\ P_{10} \\ R_{20} \\ P_{20} \end{pmatrix} \tag{11}$$

A time evolution matrix $\mathcal{C}$ for our solutions in (9) is given by

$$\mathcal{C} \equiv \begin{pmatrix} f_1 & h_1 & f_3 & g_3 \\ f_2 & h_2 & f_4 & h_4 \\ f_3 & h_3 & f_1 & g_1 \\ f_4 & h_4 & f_2 & g_2 \end{pmatrix}, \tag{12}$$

where $f_{2j-1} \equiv h_{2j} - 2\dot{h}_{2j}(0)g_1$ and $f_{2j} \equiv \dot{f}_{2j-1}$ for $j = 1, 2$.

It is convenient for our purpose to introduce the Wigner distribution function as

$$
\begin{aligned}
& W(R_1, P_1, R_2, P_2) \\
& = \frac{1}{(2\pi)^2} \int d^2r \rho \left(R_1 - \frac{r_1}{2}, R_2 - \frac{r_2}{2}, R_1 + \frac{r_1}{2}, R_2 + \frac{r_2}{2} \right) e^{i \sum_{j=1}^{2} P_j r_j}.
\end{aligned}
$$

The characteristic function [25] for the Wigner distribution is given by

$$
\chi_W(\mathcal{Y}, t) = \text{Tr} \left[\rho(0) e^{i \sum_{j=1}^{4} Y_j X_j(t)} \right], \tag{13}
$$

where we defined $\mathcal{X} \equiv (X_1 \ldots X_4)$ with $X_{2j-1} \equiv \sqrt{\Omega} R_j$, $X_{2j} \equiv P_j/\sqrt{\Omega}$ for $j = 1, 2$ and $\mathcal{Y} \equiv (Y_1 \ldots Y_4)$. We will fix the normalization for each component that appeared in $\mathcal{C}$ accordingly. The symmetric correlations can be obtained from $\chi_W(\mathcal{Y}, t)$ as

$$
\langle \{X_i, X_j\} \rangle = -\frac{\partial^2 \chi_W(\mathcal{Y}, t)}{\partial Y_i \partial Y_j} |_{\mathcal{Y}=0}, \tag{14}
$$

where $\{A, B\} \equiv (AB + BA)/2$ is an anticommutator. We trace out the field ϕ in order to obtain the reduced dynamics of the system. With a factorized initial condition: $\rho(0) = \rho_S(0) \otimes \rho_\phi(0)$, χ_W is also factorized to two components as $\chi_W(\mathcal{Y}, t) = \chi_W^S(\mathcal{Y}, t) \chi_W^\phi(\mathcal{Y}, t)$. In our case, the system part is $\chi_W^S(\mathcal{Y}, t) = \text{Tr}_S \left[\rho_S(0) e^{i \sum_{j=1}^{4} Y_j X_{Cj}(t)} \right]$, where $X_{Cj}(t)$ are solutions of Heisenberg equations with $\phi = 0$. The field characteristic function in (13) can also be evaluated exactly. We assume that environment is initially in a thermal state with an inverse temperature $\beta \equiv 1/T$. Its density matrix is given as $\rho_\phi(0) = \sum_k e^{-\beta \omega_k} | k \rangle \langle k |$. We obtain

$$
\begin{aligned}
\chi_W^\phi(\mathcal{Y}, t) &= \exp \left[-\frac{1}{2} \mathcal{Y}^T \Sigma \mathcal{Y} \right] \\
&= \exp \left[-\frac{1}{2} (Y_1 \ldots Y_4)^T \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{14} \\ & \cdots & \\ \Sigma_{41} & \cdots & \Sigma_{44} \end{pmatrix} \begin{pmatrix} Y_1 \\ \cdots \\ Y_4 \end{pmatrix} \right]. \tag{15}
\end{aligned}
$$

Here

$$
\begin{aligned}
\Sigma_{jl}(t) = \frac{\lambda^2}{2\pi} \int_0^\infty d\omega \omega e^{-\omega/\Lambda} \coth(\beta\omega/2) \int_0^t ds \int_0^t ds' \\
\times g_{\tilde{j}}(t - s) \cos \omega(s - s') g_{\tilde{l}}(t - s'), \tag{16}
\end{aligned}
$$

where $\tilde{j} \equiv (3 + (-1)^j)/2$, are time-dependent (non-equilibrium) fluctuations of the system variables, a part induced from environment. Here we introduced the cutoff frequency Λ for the field modes. Note that off-diagonal correlations Σ_{jl} for $j \neq l$

are non-vanishing in general, that is, an interaction with a common environment induces an effective interaction between two oscillator variables and thus correlation and entanglement between them.

For an initial Gaussian state with vanishing mean positions and momentums, $\langle \mathcal{X}(0) \rangle = 0$, the system characteristic function also takes a Gaussian form:

$$
\begin{aligned}
\chi_W^S(\mathcal{Y}, t) &= \exp\left[-\frac{1}{2}\mathcal{Y}^T(\Delta\mathcal{X})_C^2(t)\mathcal{Y}\right] \\
&\equiv \exp\left[-\frac{1}{2}(Y_1 \ldots Y_4)^T \begin{pmatrix} \langle\{X_{1C}, X_{1C}\}\rangle & \cdots & \langle\{X_{C1}, X_{C4}\}\rangle \\ & \cdots & \\ \langle\{X_{C4}, X_{C1}\}\rangle & \cdots & \langle\{X_{C4}, X_{C4}\}\rangle \end{pmatrix} \begin{pmatrix} Y_1 \\ \ldots \\ Y_4 \end{pmatrix}\right],
\end{aligned}
$$
(17)

where $\mathcal{X}_C = (X_{C1} \cdots X_{C4})$ satisfy the equations of motion (8) for damped harmonic oscillators. $(\Delta\mathcal{X})_C^2(t)$ are essentially the initial fluctuations of system variables shifted by damped oscillatory motion of a coupled harmonic oscillator. Combining with the characteristic function for the field, we obtain

$$
W(\mathcal{X}, t) = \frac{1}{(2\pi)^2} \frac{1}{(\det(\Delta\mathcal{X})^2(t))^{1/2}} \exp\left[-\frac{1}{2}\mathcal{X}^T((\Delta\mathcal{X})^2(t))^{-1}\mathcal{X}\right],
$$

where $(\Delta\mathcal{X})^2 = (\Delta\mathcal{X})_C^2 + \Sigma$.

3 Entanglement dynamics of quantum Brownian oscillators

Let us consider a two-mode squeezed state with a squeezing parameter r as an initial state [26]. Its correlation matrix is

$$
(\Delta\mathcal{X})_C^2(0) \equiv \langle\{\mathcal{X}_C(0), \mathcal{X}_C^T(0)\}\rangle = \frac{1}{2}\begin{pmatrix} \cosh(2r)\mathbf{1} & -\sinh(2r)\sigma_3 \\ -\sinh(2r)\sigma_3 & \cosh(2r)\mathbf{1} \end{pmatrix}.
$$

In the Wigner representation, the same state can be expressed as

$$
W(R_1, R_2, P_1, P_2) = \frac{4}{\pi^2} e^{-e^{2r}\left[\Omega(R_1-R_2)^2 + (P_1+P_2)^2/\Omega\right] - e^{-2r}\left[\Omega(R_1+R_2)^2 + (P_1-P_2)^2/\Omega\right]}.
$$
(18)

This state can be obtained by acting a squeezing operator $e^{ir(R_1 P_2 - P_1 R_2)}$ on the vacuum.

A criteria for separability of a bipartite two-level-system was studied in [13]. The necessary and sufficient condition for separability of the density matrix is to have only non-negative eigenvalues after partial transpose of one of its subsystem. The same criteria is not always sufficient for the bipartite system with more than two levels [14]. The extension of this criteria to continuous Gaussian variables was first given in [15]. For Gaussian variables, the partial transpose of a density matrix in a coordinate representation for one oscillator component is equivalent to a mirror reflection of that

component in the Wigner distribution function. The necessary and sufficient condition for a continuous variable quantum state to be separable is that the partial mirror reflected state is still a physical quantum state that satisfies the uncertainty principle. In the phase space representation, the partial mirror reflection on the second variable can be stated as $(R_1, P_1, R_2, P_2) \rightarrow (R_1, P_1, R_2, -P_2)$. In terms of $\mathcal{X}$, this can be expressed as the matrix operation by the matrix $\eta \equiv \mathrm{diag}(1, 1, 1, -1)$ as $\mathcal{X} \rightarrow \eta \mathcal{X}$. It follows that the partial mirror reflection transforms the covariance matrix as

$$(\Delta \mathcal{X})^2 \rightarrow \eta (\Delta \mathcal{X})^2 \eta^T. \tag{19}$$

Before we apply the above criteria to our covariance matrix $(\Delta \mathcal{X})^2$, let us make three local Bogoliubov transformations to simplify the form of the covariance matrix. These local transformations do not change the separability of the system. First we consider the local orthogonal transformation

$$M_4 \equiv \begin{pmatrix} M_2 & 0 \\ 0 & M_2 \end{pmatrix} \in O(2, R) \bigotimes O(2, R) \subset O(4, R), \tag{20}$$

where $M_2 \in O(2, R)$ is an orthogonal matrix. Under M_4, a symmetric matrix of the form

$$(\Delta \mathcal{X})^2 \equiv \langle \{\mathcal{X}, \mathcal{X}^T\} \rangle \equiv \begin{pmatrix} D & A \\ A^T & D \end{pmatrix} \tag{21}$$

transforms to

$$M_4 (\Delta \mathcal{X})^2 M_4^T \equiv \begin{pmatrix} M_2 D M_2^T & M_2 A M_2^T \\ M_2 A^T M_2^T & M_2 D M_2^T \end{pmatrix}. \tag{22}$$

By a suitable choice of M_4, we can diagonalize D. Next we make a local symplectic transformation with another matrix $S_4 \in Sp(2, R) \bigotimes Sp(2, R)$ that has a form:

$$S_4 \equiv \begin{pmatrix} S_2 & 0 \\ 0 & S_2 \end{pmatrix}, \tag{23}$$

where $S_2 \in Sp(2, R)$ is a symplectic matrix. For a suitable choice of S_2, we can make $S_2 M_2 D M_2^T S_2^T$ to be diagonal with an equal component d. Furthermore another transformation with an orthogonal matrix $O_4 \in O(2, R) \bigotimes O(2, R)$ that has a form:

$$O_4 \equiv \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \tag{24}$$

can make $(\Delta \mathcal{X})^2$ into the following canonical form:

$$(\Delta \mathcal{X}_C)^2 = \begin{pmatrix} d & 0 & a & 0 \\ 0 & d & 0 & b \\ a & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}. \tag{25}$$

From the Williamson's theorem [27], there exists a symplectic transformation that diagonalizes a positive-definite 4×4 symmetric matrix into the following form:

$$(\Delta \mathcal{X}_D)^2 = \begin{pmatrix} \zeta_1 & 0 & 0 & 0 \\ 0 & \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_2 \end{pmatrix}.$$

Although such a symplectic transformation does not preserve the eigenvalue spectrum in general, the diagonal components ζ_l for $l = 1, 2$ can be calculated. Writing a commutation relation in a matrix form as $\left[X_i, X_j \right] = i \Gamma_{ij}$ with

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

we construct a real symmetric matrix $\Delta \mathcal{X} \Gamma (\Delta \mathcal{X})^2 \Gamma^T \Delta \mathcal{X}$. This matrix has an eigenvalue spectrum ζ_l^2 ($l = 1, 2$) [28]. The uncertainty relation can be generalized to a symplectic invariant form $(\Delta \mathcal{X})^2 + i \Gamma / 2 \geq 0$. By changing to the diagonalized form $(\Delta \mathcal{X}_D)^2$ above, the uncertain relation is equivalent to saying that $\zeta_l \geq 1/2$ for all l. Although this uncertain relation is invariant in arbitrary symplectic transformations, they can change the entanglement property. Thus we restrict our transformation to local symplectic transformations and use the canonical form of $(\Delta \mathcal{X}_C)^2$ in (25) for our analysis. Our separability criteria is invariant under local transformations.

The uncertainty relation expressed by the components in $(\Delta \mathcal{X}_C)^2$ are given by

$$\begin{aligned} (d + a)(d + b) &\geq \tfrac{1}{4} \\ (d - a)(d - b) &\geq \tfrac{1}{4} \end{aligned} \tag{26}$$

These are generalizations of the familiar uncertain relations for pure state two oscillators to general mixed states. They can be expressed as $\Delta \tilde{R}_1^2 \Delta \tilde{P}_1^2 \geq \tfrac{1}{4}$ and $\Delta \tilde{R}_2^2 \Delta \tilde{P}_2^2 \geq \tfrac{1}{4}$ in the coordinates that diagonalize the correlation matrix $(\Delta \mathcal{X})^2$.

Under the partial transpose in (19), $b \to -b$. Thus the conditions for separability can be written as

$$\begin{aligned} (d + a)(d - b) &\geq \tfrac{1}{4} \\ (d - a)(d + b) &\geq \tfrac{1}{4}. \end{aligned} \tag{27}$$
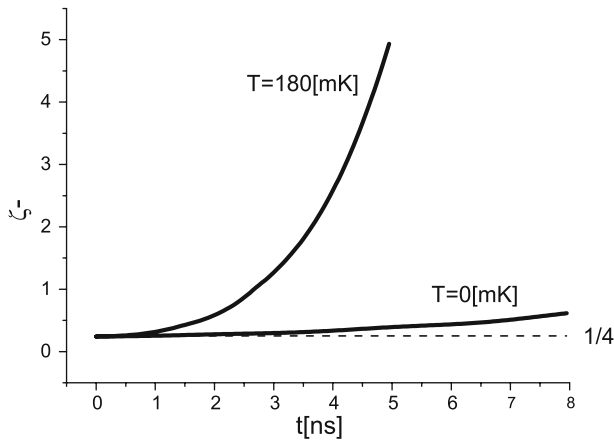
Fig. 1 The temporal evolution of the uncertainty function (ζ_- in Eq. 28) is plotted. The initial condition is a two mode squeezed state with $r = 0.05$. Other parameters are $\gamma = 0.01\,\text{GHz}, \Lambda = 50\,\text{GHz}, \Omega = 1.0\,\text{GHz}$

One can easily see that the two-mode squeezed state we introduced in (18) satisfies the uncertainty relation in Eq. 26. The separability condition (27) implies $e^{4r} \geq 1$ and $e^{-4r} \geq 1$, which only holds if $r = 0$. Thus this state is entangled as long as $r \neq 0$. In $r \to \infty$ limit, the state becomes the original EPR state discussed in [9].

In order to make invariance under the local Bogoliubov transformation manifest, one can write uncertainty relations (26) explicitly by using the symplectic invariants constructed from the determinants of covariances $|A|, |D|, |(\Delta \mathcal{X})^2|$ as

$$\zeta_\pm^2 = |D| + |A| \pm \sqrt{(|D| + |A|)^2 - |(\Delta \mathcal{X})^2|} \geq \tfrac{1}{4}. \qquad (28)$$

In Fig. 1, the temporal behavior of the uncertainty function ζ_- is plotted. The initial state is a pure two mode squeezed state and satisfies the minimum uncertainty $1/4$. As the state becomes mixed, the uncertainty increases monotonically in time even for a zero temperature case. At higher temperature, the rate of increase is faster.

Similarly the separability conditions (27) are

$$\lambda_\pm^2 = |D| - |A| \pm \sqrt{(|D| - |A|)^2 - |(\Delta \mathcal{X})^2|} \geq \tfrac{1}{4}. \qquad (29)$$

Note that the inequalities for ζ_+ and λ_+ in (28) and (29) follow automatically from those for ζ_- and λ_-. Thus λ_- carries the essential information on the separability of quantum states. In Fig. 2, time evolution of the λ_- is plotted. For an initial coherent state, the uncertainty relation for the partial transposed state is always satisfied throughout the whole evolution indicating that there is no entanglement. For an initial squeezed state ($r = 0.1$), the uncertainty relation is violated initially but eventually satisfied indicating that there is a crossover from an entangled to a separable state. The asymptotic value of separability seen in λ_- appears to be independent of the degree of initial squeezing.
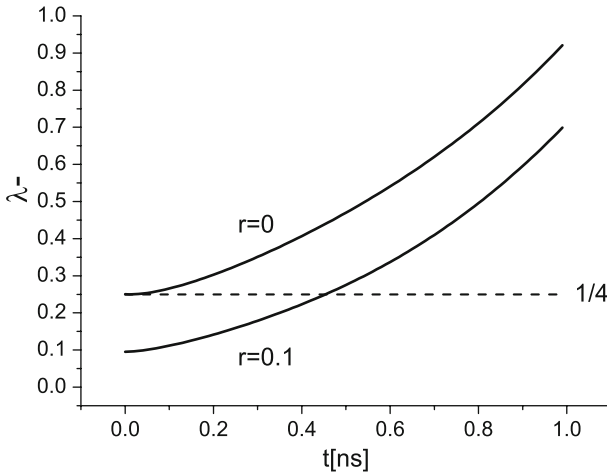
Fig. 2 The temporal evolution of the uncertainty function (λ_- in Eq. 29) after partial transpose is plotted. The initial condition is a two mode squeezed state with $r = 0.1$ for the thick solid curve and a coherent state ($r = 0$) for the thin solid curve. Other parameters are $\gamma = 0.1$ GHz, $\Lambda = 50$ GHz, $\Omega = 1.0$ GHz, $T = 0$

The negativity $\mathcal{N}$ [16,29] for quantum Brownian particles can be defined as

$$\mathcal{N} = \frac{||\rho_r^T|| - 1}{2}, \tag{30}$$

where ρ_r^T is the reduced density matrix after partial transpose. $\mathcal{N}$ is equal to the sum of all negative eigenvalues of ρ_r^T and measures how much ρ_r^T fails to be positive. From the Peres criteria, it can be used as a measure of entanglement. It also has a nice property as an entanglement monotone such that it does not increase under local operations and classical communications. The logarithmic negativity $E_\mathcal{N}$ defined as

$$E_\mathcal{N} = \log_2 ||\rho_r^T|| \tag{31}$$

also has the similar property. Since diagonalization of $(\Delta \mathcal{X})^2$ brings the original state into the thermal state, the partial transposed density matrix after the same transformation also has the thermal form that can be written as a function of the symplectic invariants $\lambda_\pm$ as

$$\rho_r^T = \prod_\pm \left[\left(\frac{2}{2\lambda_\pm + 1} \right) \sum_{n=1}^{\infty} \left(\frac{2\lambda_\pm - 1}{2\lambda_\pm + 1} \right)^n |n_\pm\rangle\langle n_\pm| \right]. \tag{32}$$

For separable states, $\lambda_\pm \geq 1/2$. Thus $||\rho_r^T|| = 1$ and $\mathcal{N} = E_\mathcal{N} = 0$ follows. For entangled states, $\lambda_- < 1/2$ but $\lambda_+ \geq 1/2$. The latter follows because $\lambda_+ > \zeta_-$ for $|A| < 0$ (if $|A| > 0$, the state is separable from (26) to (29)). $||\rho_r^T|| = 1/2\lambda_-$ follows. Thus both $\mathcal{N}$ and $E_\mathcal{N}$ can be expressed in terms of λ_- as
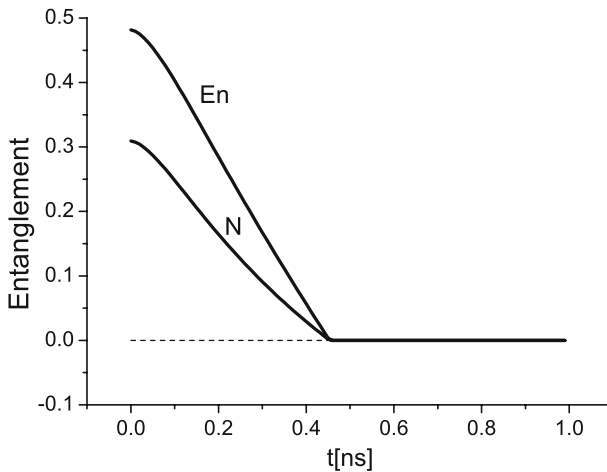
Fig. 3 The negativity $\mathcal{N}$ and the logarithmic negativity $E_\mathcal{N}$ are shown. The initial state is a two-mode squeezed state with $r = 0.1$. $\mathcal{N}$ and $E_\mathcal{N}$ both remain zero in case of the two mode coherent state initial condition $r = 0$. Other parameters are $\gamma = 0.1$ GHz, $\Lambda = 50$ GHz, $\Omega = 1.0$ GHz, $T = 0$

$$\mathcal{N} = \max \left[0, \frac{1 - 2\lambda_-}{4\lambda_-} \right],$$

$$E_\mathcal{N} = \max \left[0, -\log(2\lambda_-) \right]. \tag{33}$$

In Fig. 3, the negativity $\mathcal{N}$ and the logarithmic negativity $E_\mathcal{N}$ are shown as a function of time. The initial pure two mode squeezed state has the maximum entanglement that decays monotonically in time. They both vanish at $t_{DE} = 0.45$ ns and remain zero. This disentanglement time scale t_{DE} measured this way is the same as the time when the uncertainty relation for the partial transposed state recovers (the lower curve in Fig. 2) as we expect.

4 Summary

In this work, we used two quantum Brownian oscillator model to study the dynamical aspect of quantum entanglement without Born–Markov approximation. We studied several different criteria for quantum entanglement. The uncertainty function for the two-mode squeezed state under partial transpose initially violates the uncertainty principle but eventually satisfies it. Invoking the Peres-Horodecki-Simon's criteria, this corresponds to the temporal crossover from an entangled to separable state. The negativity and the logarithmic negativity show a monotonic decrease and vanish indicating the similar crossover. We thus saw that, through the analysis of exact dynamics, the effect of environment destroys quantum entanglement among Brownian oscillators through the decoherence mechanism.

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